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An Eulerian Method for Transient Nonlinear Free Surface Wave Problems

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An Eulerian difference method is developed for the transient potential flow of an incompressible fluid with fully nonlinear free surface conditions. The free surface coordinate $y = \eta(x, t)$ and the velocity potential $\phi(x, y = \eta; t)$ on the free surface are recognized as the primary unknowns to be solved as an initial value problem from the pair of nonlinear partial differential equations representing the dynamic and the kinematic conditions of the free surface. The continuity relation $\nabla^2\phi = 0$ for the velocity potential $\phi(x, y; t)$ over the flow field Ω below the free surface is recognized as a subsidiary condition to be enforced at all times. The field of computation is transformed into a time invariant cartesian region with the free surface $\eta(x, t)$ represented by a coordinate line (or surface). The iterative solution for $\phi(x, y; t)$, $p(x; y; t)$ in this fixed field of computation is facilitated by the use of Fast Fourier Transform (FFT). The iterative process converges rapidly. In terms of this converged $\phi(x, y; t)$,the free surface location $\eta(x, t)$ and its potential $\phi(x, y = \eta; t)$ are advanced in time. Results from two planar examples are illustrated. The method is equally applicable to problems in three space dimensions, possibly involving interactive matching with neighboring flow fields. If the initial free surface potential $\phi(x, y = \eta; t = 0)$ is unknown, difficulties may be encountered in data specification for securing a well posed problem for solution.

INTRODUCTION

Free surface wave problems are characterized by the presence of some unknown, time dependent free surface of a fluid. Various methods have been developed to solve such problems. The potential flow problems of an incompressible fluid in two space dimensions have been dealt with quite succesfully with the harmonic flow field underneath the free surface represented by some unknown analytic complex function, i.e., the complex potential $\phi + i\psi$, where ϕ is the velocity potential and ψ the stream function. Such a complex representation greatly facilitates the evaluation of the Green's function that permits the expression of the solution of the harmonic field through boundary perturbations and their associated Green's functions. Then the dynamic relation, defining the evolution of the free surface stands as an integro-differential equation that may be solved iteratively. When linearized, the solution can also be effected through normal mode analysis. The temporal evolution of the

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free surface have thus been solved numerically for selected configurations [1–4]. Despite some numerical difficulties, such solutions enable us to learn much about the free surface waves, with Lagrangian Marker Particles providing vivid displays of the evolution.

The methods described above are, however, not useful for problems in three space dimensions without a convenient extension of the complex representation of the harmonic fields to facilitate the evaluation of the Green's functions. The nontrivial boundaries of problems of practical interest introduce additional difficulties. There have therefore been considerable efforts in developing computational methods for the solution of the free surface problems directly from the Euler or the Navier–Stokes system of equations with nonperiodic and/or otherwise nontrivial boundaries. Hopefully, such computational methods will be equally applicable to free surface problems in three as well as two space dimensions. Many such computational methods have been developed in which the flow conditions on the free surface are linearlized and the free surface location interpolated. The computational process is complicated with result yet to be desired.

The MAC methods [5–8] and their modified versions [9–10] are popular. They employ some finite difference method to advance the flow fields at a given time over a fixed Eulerian set of cells (or grids). The free surface is outlined by those cells with scarcely few (or none) Marker Particle(s). Marker Particle(s) is assigned in each cell to indicate the presence of fluid in the cell, and its location is advanced by the local fluid velocity at successive time steps. This Lagrangian up-dating of Marker Particles near the free surface determines, according to some preconceived format, the new free surface at the advanced times. Such a mixed Eulerian–Lagrangian computation is tedious but intuitively appealing. Their global results are often impressive despite compromising details.

Finite element methods have also been developed [11–16] for such problems. In view of the changing free surface at successive time steps, the grids must be rezoned repeatedly; and the discrete formulation in terms of the set of finite elements over the new grids and of the changing free surface boundary conditions must be altered at every time step. These finite element methods are as tedious as the MAC methods. Globally acceptable results can also be obtained if computational instability can be avoided or suppressed.

The moving free surface as a boundary of the computational region is widely recognized as a major source of difficulty(ies). A moving boundary can, however, be simply removed by using the unknown free surface as a coordinate (at least for the nonbreaking free surface) in the transformed computational space. The physical location of the free surface can be recovered through inverse transformation after the solution has been obtained in the transformed space. Thus for a potential flow problem, the finite difference (or element) solution of the Laplace equation for the velocity potential can be carried out in the transformed space with proper boundary formulation. The solution in the physical space is then obtained through inverse mapping [17–19]. With appropriately linearized boundary formulation such solution at fairly coarse meshes yields reasonably looking results, although

computational instability often appears at more refined meshes. When the fully nonlinear boundary formulation on the free surface is adopted, instability becomes more severe so that some "filtering" scheme (or other forms of computational artifices) has to be introduced to restore results comparable to those with linearized boundary formulations. It is apparent that computational difficulty does not lie entirely in the unknown, moving free surface boundary.

Physically, the dynamic and the kinematic statements for the motion of fluids on and near the free surface define the temporal development of the local fluid flow $\bar{q}_t$ and η_t (i.e., the time derivatives of fluid velocity and the free surface location respectively). They are wave equations, defining the course of temporal development of surface waves; not field equations defining the spatial variation of velocity potential at any time t as boundary condition for the determination of the flow field Ω, below the free surface. To use the dynamic relation as such boundary conditions, one has to approximate $\bar{q}_t$ and/or η_t. When the flow field at the advanced time step is determined, the condition of zone of dependence of the surface wave development may have been violated with consequent computational instability.

The MAC methods treat the entire field as a wave problem with the fluid velocity $\bar{q}$ advanced in time as an initial value problem via pressure and/or vorticity, supposedly determined by the subsidiary relation of continuity that the velocity field must remain solenoidal at all times. This is easier said than done, and often has to be compromised along with some approximate treatment of the free surface location and the free surface conditions. With appropriate details, it is possible to generate globally reasonable results. We are, however, disturbed not only by the various comprising details, but also by the failure of the calculated velocity field to become reasonably solenoidal. Moreover, such treatments of initial value problems with the infinite signal speeds inevitably violates the condition of zone of dependence at any finite time step. Computational difficulty is thus to be expected. Physically the free surface wave under gravity is essentially "frozen" while the pressure disturbances within the incompressible fluid are relaxed "instantaneously" by the "infinitely" fast pressure waves to restore the solenoidal velocity field $\nabla \cdot \bar{q} = 0$. For each computational time step, selected to correspond to the characteristic time of the surface waves, the velocity $\bar{q}$ in Ω should be determined at any instant t from the solenoidal condition with fixed boundary, not up-dated as in the MAC methods through momentum equations. The velocity field $\bar{q}$ should be solenoidal and the pressure field should be smooth at all times without filtering and smoothing.

The disparity of the speeds of propagation of the pressure waves in the fluid interior and the gravitational wave on the fluid surface suggests the following computational approach: The dynamic and the kinematic relations for the fluid on the free surface are treated as wave equations for the temporal development of free surface locations and of fluid velocity on it. The velocity $\bar{q}$ in Ω, below the free surface is determined at every instant by fixed boundary value problems on Ω. Since surface dynamics is coupled to the solution of $\bar{q}$ in the field Ω, some iterative method of solution is generally required. The iterative sequence of the velocity field $\bar{q}$ must

be solenoidal so that the iterative solution of the surface wave equation is carried out in the solenoidal subspace. To demonstrate the physical concept, we give in the next section the mathematical formulation of the potential flow problem in two space dimensions. Its generalization to three space and more complex cases is straightforward.

FORMULATION

Consider here the transient potential flow of an incompressible fluid with velocity $\bar{q} = \nabla\phi$ in terms of the velocity potential function $\phi(x, y; t)$ in two space dimensions, initiated by some pressure disturbance on the free surface. The free surface at any instant t is defined by $y = \eta(x, t)$, with the gravitational acceleration g acting in the negative y direction and $\eta(x, t)$ some single valued function of x at any t. (Thus the present consideration is limited to free surface that does not break in its course of development.) Here (x, y) designates a rectangular cartesian coordinate system with origin fixed at some convenient point at all times. The atmospheric pressure acting on the free surface is p_0. We hope to determine the potential function $\phi(x, y; t)$ and the pressure $p(x, y; t)$ in the region Ω below the free surface $\eta(x, t)$ at all times $t > 0$. For simplicity, we take the region Ω to be a two dimensional tank bounded by rigid, impermeable solid walls at $x = \pm L/2$ and a flat bottom $y = -d$. $y = 0$ represents the free surface when the fluid is in static equilibrium. Let the fluid density be ρ and the representative velocity of the ensuing fluid motion be U. Define the Froude Number $F_r = U/(gL)^{1/2}$ and the free surface pressure parameter $\sigma = p_0/\rho g L$. Then the momentum equation for such a potential flow field becomes

$$\nabla\left[\phi_t + \frac{q^2}{2} + \frac{\sigma p}{F_r^2} - \frac{y}{F_r^2}\right] = 0. \tag{1}$$

The quantity in the square bracket is a function of time only, i.e., the Bernoulli's integral for such potential flows. If the dimensionless pressure on the free surface $y = \eta(x, t)$ is taken to be zero, we have the dynamic relation along $\eta(x, t)$ as

$$\phi_t = -\frac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{1}{F_r^2}\eta(x, t) \tag{2}$$

where $\eta(x, t)$ defines the instantaneous free surface, satisfying the kinematic condition

$$\eta_t = \phi_y - \phi_x \eta_x. \tag{3}$$

Here ϕ_x and ϕ_y are the partial derivatives of $\phi(x, y; t)$ with (y, t) and (x, t) held constant, respectively, and evaluated at x, $y = \eta(x, t)$, i.e., on the free surface. This $\phi(x, y; t)$ is the harmonic velocity potential, to be determined from the continuity relation

$$\nabla \cdot \bar{q} = \nabla_{(x,y)}^2 \phi = 0 \tag{4}$$

under properly posed boundary data over the region $\Omega(x, y; t)$. Thus Eqs. (2) and (3) stands as the pair of wave equations governing the temporal development of the free surface $\eta(x, t)$ and the velocity potential $\phi[x, \eta(x, t); t]$ on the free surface. Equation (4) is a subsidiary condition that defines the harmonic function ϕ and its spatial derivatives ϕ_x and ϕ_y that are needed in Eqs. (2) and (3). As a velocity potential in Ω, $\phi(x, y; t)$ will provide the fluid velocity in Ω as

$$\bar{q}(x, y; t) = \nabla\phi(x, y; t). \tag{5}$$

Differentially speaking, $\bar{q}$ as defined by (5) is solenoidal as ϕ satisfies (4). This is not necessarily true if ϕ and η are known or defined only as netfunctions in discrete form. It is important to be assured of the solenoidal property of the velocity field if the pressure field $p(x, y; t)$ is to be evaluated accurately as an integral of Eq. (1) with $p_0 = 0$:

$$p = \sigma^{-1}[y - F_r^2\{\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\}]. \tag{6}$$

Given a set of initial data $\eta(x, t=0)$ and $\phi[x, \eta(x, 0); t=0]$, we have to solve first Eq. (4) to obtain $\phi(x, y; t=0)$ over $\Omega(x, y; t=0)$ and then evaluate ϕ_x and ϕ_y on $y = \eta(x; t=0)$ to permit the evaluation of ϕ_t and η_t from Eqs. (2) and (3). In some discretized form, $\eta(x, \Delta t)$ and $\phi[x, \eta(x, \Delta t); \Delta t]$ at the advanced time step Δt can then be solved, provided that the condition of zone of dependence (or the CFL condition in linearized form) of the free surface wave is obeyed. Details of implementing the discrete solution are illustrated in the next section.

We note that, at any time step, the solution of $\phi(x, y; t)$ from Eq. (4) under the "fixed" free surface location $\eta(x, t)$ is "linear" and effectively "uncoupled" from the nonlinear problem of advancing the free surface wave. Various useful superposition techniques can be employed to reduce the solution of the Poisson equation to standard component forms, which are given in the next section. The asymptotic concept of uncoupling the incompressible flow field below the free surface from the free surface dynamics itself is generally valid, not only for more complicated initial value problems of the Poisson equation, but also for rotational and/or stratified flow fields in three as well as two space where the Poisson equation will be replaced. The illustrative examples given below are simple to avoid possible confusion of the fundamental issue with the computational details of more complicated examples.

Computational Solution

Introduce the coordinate transformation

$$\xi = \frac{y + d}{\eta + d} \tag{7}$$

so as to map the time varying region $\Omega(x, y; t)$ into a time invariant rectangular region of computation $-1/2 \leqslant x \leqslant 1/2$; $0 \leqslant \xi \leqslant 1$, (Figs. 1a, b). In terms of the trans-

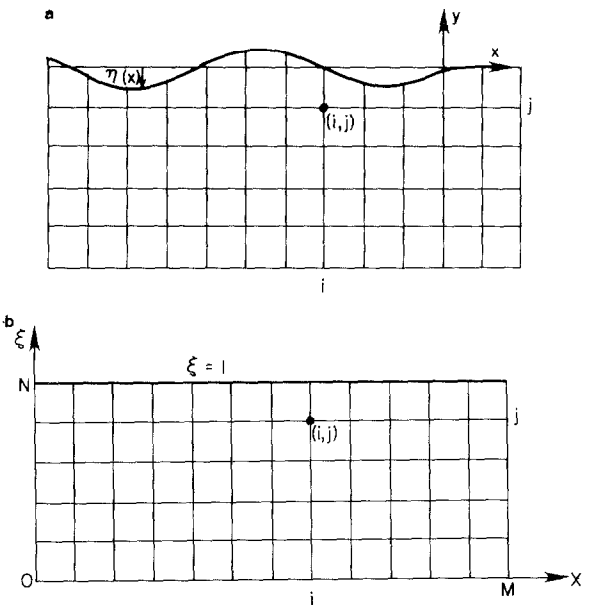

FIG. 1. (a) Flow field below free surface $\eta(x, t)$ to be computed at a given time t. (b) Transformed computational demain with fixed boundary at free surface $\xi = 1$.

formed variable (x, ξ), the Laplace equation (4) becomes (8a). The wave equation (2) and (3) become (8b) and (8c). The boundary data for $\phi(x, y; t)$ become (8d) and (8e)

$$\phi_{xx} + \phi_{\xi\xi} = \left[1 - \frac{1 - \xi^2 n_x^2}{(\eta + d)^2}\right] \phi_{\xi\xi} - \frac{2\eta_x^2 - \eta_{xx}(\eta + d)}{(\eta + d)^2} \xi \phi_\xi + \frac{2\xi \eta_x}{\eta + d} \phi_{x\xi} \qquad \text{in } \Omega(x, \xi) \tag{8a}$$

$$\phi_t = -\frac{1}{2}\left(\phi_x - \phi_\xi \frac{\eta_x}{\eta + d}\right)^2 - \frac{1}{2}\phi_\xi^2/(\eta + d) - \frac{1}{F_r^2} n \qquad \text{on } \xi = 1 \tag{8b}$$

$$\eta_t = \frac{\phi_\xi}{\eta + d} - \eta_x \left(\phi_x - \frac{\eta_x}{\eta + d}\phi_\xi\right) \qquad \text{on } \xi = 1 \tag{8c}$$

$$\phi_x = \frac{\xi \eta_x}{\eta + d} \phi_\xi \qquad \text{on } x = 0, 1 \tag{8d}$$

$$\phi_\xi = 0 \qquad \text{on } \xi = 0. \tag{8e}$$

The initial data for ϕ and η on $\xi = 1$, at $t = 0$ are:

$$\phi(x, \xi = 1; t = 0) = g(x) \tag{8f}$$

$$\eta(x; t = 0) = \eta_0(x) \tag{8g}$$

With $\eta(x)=\eta_0(x)$ specified, Eq. (8a) is first solved with the boundary data (8d), (8e) and (8f) in (x, ξ) space. We introduce an iterative process to solve Eq. (8a) as a Poisson equation for the advanced iterate $\phi^{(k+1)}$ with its source terms on the right-hand side of (8a) evaluated with the known previous iterate $\phi^{(k)}$. The iterative process is considered to be converged when $\|\phi^{(k+1)}-\phi^{(k)}\|_\infty < \varepsilon \simeq 10^{-5}$. The differential operators are discretized with simple centered spatial differences to give the system of difference equations:

$$\phi(i-1, j)+\phi(i+1, j)+\alpha^2\phi(i, j-1)+\alpha^2\phi(i, j+1) \\ -2(1+\alpha^2)\,\phi(i, j)=\Delta x^2 \cdot R(i, j) \tag{9}$$

where $\alpha=\Delta x/\Delta\xi$ and $\phi(i, j)$ represents the $(k+1)$th iterate of ϕ evaluated at $i\Delta x$ and $j\Delta\xi$ over all the grid points interior of Ω; i.e., $0\leqslant i\leqslant M$, $0\leqslant j\leqslant N$ with $M\Delta x=1$ and $N\Delta\xi=1$. $R(i, j)$ represents the right-hand side of Eq. (8a) evaluated with the known kth iterate of ϕ.

For the Neumann boundary condition on a boundary point $i=0$ or M, such as (8d), the normal derivative can be expressed in terms of the current values of ϕ on the grid points in the immediate neighborhood, while the nonzero value of the normal derivative is evaluated in terms of the previous iterate to be combined with Δx^2 $R(i, j)$. Then Eq. (9) becomes

$$\phi(i-1, j)+\phi(i+1, j)+\alpha^2\phi(i, j-1)+\alpha^2\phi(i, j+1) \\ -2(1+\alpha^2)\,\phi(i, j)=B(i, j) \tag{10}$$

with $0\leqslant i\leqslant M$, $0\leqslant j\leqslant N$. Thus equations system (10) stands as a matrix equation of order $(M)\times(N)$ with tridiagonal bands for the solution of all $\phi(i, j)$ interior of Ω as if all the normal derivatives on the Neumann boundaries are zero. Introduce the appropriate Fourier transform along the x-direction which, for the present application, is

Synthesis

$$\phi(i, j)=\sum_{s=0}^{M} E(s, M)\,\phi_s(j)\cos\left(\frac{\pi i s}{M}\right) \qquad (0\leqslant i\leqslant M) \tag{11a}$$

Analysis

$$\phi_s(j)=\frac{2}{M}\sum_{i=0}^{M} E(i, M)\,\phi(i, j)\cos\left(\frac{\pi s i}{M}\right) \qquad (0\leqslant s\leqslant M) \tag{11b}$$

with $E(s, M)=1/2$ if $s=0$ or M and $=1$ otherwise. A similar decomposition of $B(i, j)$ into $B_s(j)$ and substitution into Eq. (10) give:

$$\alpha^2\phi_s(j-1)+2\left[\cos\frac{\pi s}{M}-(1+\alpha^2)\right]\phi_s(j) \\ +\alpha^2\phi_s(j+1)=B_s(j) \qquad \text{with} \quad 0\leqslant s\leqslant M. \tag{12}$$

Equations system (12) for each Fourier component, ϕ_s, is tridigonal. Thus when the free surface data $\phi(x, \xi=1; t=0)=g(x)$ is resolved into its Fourier components $g_s(N)$ as is in Eq. (11b) and when the Neumann condition on the bottom wall $j=0$ is incorporated, all $\phi_s(j)$ with $0\leqslant j\leqslant N$ can be explicitly solved from (12). This operation is repeated for all $0\leqslant s\leqslant M$ to obtain all $\phi_s(j)$ in the interior of Ω. Synthesis of $\phi_s(j)$ over s according to equation (11a) yields the desired solution $\phi(i, j)$ or $\phi(x, \xi; t)$ for the specific boundary data.

We are now ready to advance $\eta(x, t)$ and $\phi(x, \xi=1; t)$ to the next step $t+\Delta t$.With the discretized forms of Eqs. (8b) and (8c),

$$\begin{aligned}\phi^{n+1}(i)&=\phi^n(i)+[F^{n+1}(i)+F^n(i)]\,\Delta t/2\\ \phi^{n+1}(i)&=\eta^n(i)+[G^{n+1}(i)+G^n(i)]\,\Delta t/2\end{aligned}\tag{13}$$

where $F(i)$ and $G(i)$ are the discretized forms (say centered spatial difference) of the right-hand sides of Eqs. (8b) and (8c). Both F and G involve ϕ_x and ϕ_ζ on $\xi=1$ to be evaluated from $\phi(x, \xi)$ synthesized from $\phi_s(j)$ that are obtained from Eq. (12).

The implicit form of Eq. (13) calls for its simultaneous or iterative solution of $\eta(x, \xi=1)$, $\phi(x, \xi=1)$ at $t+\Delta t$, (i.e., the $n+1$ time step). When satisfactory convergence is obtained, we proceed to the next time step, i.e., to solve the boundary value problem of Poisson equation (8a) for the potential $\phi(x, \xi, t+\Delta t)$.

The pressure field $p(x, \xi; t)$ or $p(x, y; t)$ within the domain Ω can be calculated directly from Eq. (6), when desired, from the converged $\phi(x, \xi; t)$. It does not participate in the solution procedure of η and ϕ.

Sample Results

Two sample computations have been carried out with Cyber 170–172 on a $18\times 18(x, \xi)$ rectangular mesh. The fast Fourier transform (FFT) was implemented with complex subroutine. The parameters of the free surface flow were chosen as,

$$d/L=1.7, \qquad F_r^2=1/2, \qquad \sigma=0.1.$$

The time step for integrating the wave equations is $\Delta t=0.02\,L/U$. Iterative convergence is considered as obtained when the relative maximum change of functions in successive iterations is $<10^{-5}$.

The first example illustrates the relaxation of a distorted free surface $\eta_0(x, t=0)=0.2\cos(\pi x)$ for positive times $t>0$. The fluid was assumed to be maintained at rest for $t<0$ by some arbitrary free surface pressure distribution which is removed or restored to the atmospheric pressure p_0 at $t=0$ so that $\phi(x, y=\eta; t=0)=0$. We wish to calculate the development of the free surface from the disturbed initial state of a $\eta_0(x, t=0)$ and $\phi_0(x, t=0)=0$. Firstly, the velocity potential $\phi(x, y; t=0)$ immediately after the restoration to the atmospheric pressure on the free surface $\eta_0(x, t)$ is calculated. Then, the free surface coordinate

$\eta(x, \Delta t)$ and the free surface potential $\phi(x, \xi = 1; \Delta t)$ at the next time step Δt are calculated. This process is repeated to obtain $\eta(x, t)$ and $\phi(x, y; t)$ at successive time steps as is described in the previous section. The pressure field $p(x, y; t)$ at any time step can be obtained directly from Eq. (6) when desired. It takes generally five iterations to obtain satisfactory convergence of ϕ and η for each time step. The free surface locations $\eta(x, t)$ at several time steps are illustrated in Fig. 2.

The second example illustrates the motion created by some suitable distribution of impulsive pressure acting on the free surface in static equilibrium $y_0(x) = 0$, so as to produce an instantaneous free surface potential

$$\phi_0(i, \xi = 1, t = 0) = \phi_0(i - 1, \xi = 1; t = 0) - 0.042 \sin(\pi x)$$

with $\phi_0(0, \xi = 1; t = 0) = 0$, and $x = i\Delta x$. The harmonic field $\phi(x, y; t = 0)$ obtained as the solution of Eq. (10) gives the initial velocity ϕ_x and ϕ_y of the fluid throughout Ω resulting from the initial impulses. The calculation proceeds in the same manner as is in the previous example and described in the previous section. For this type of problem it requires ten iterations for the solution of Eqs. (10) and (13) to obtain the converged results of $\eta(x)$ and $\phi(x, \xi)$ for each time step. The results are illustated in Fig. 3.

The two examples given above represent the computation of a single Fourier component of any general Fourier decomposable initial data of $\eta(x, t = 0)$ and $\phi(x, y = \eta;\ t = 0)$. The computational solution of problems with such a general initial-boundary data involves simply the synthesis of the results of various components illustrated above at each time step prior to the iterative solution of the nonlinear wave problem. Besides additional computational effort, we do not anticipate any fundamental difficulty.

The computational formulation is efficient. For each time step, there are two iterative loops. Loop (a) is needed for the correction of the source terms in the solution of the potential function ϕ from Poisson equation 8(a). Loop (b) is needed for the quasi-linear up-dating of the nonlinear surface wave according to the

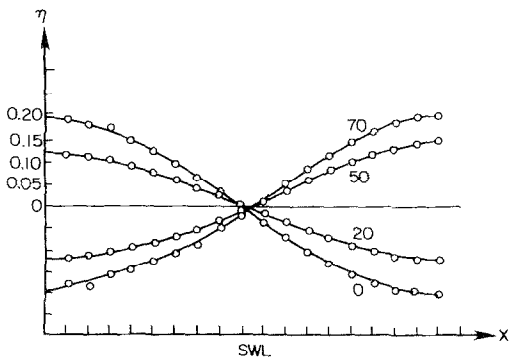

FIG. 2. Evolution of a distorted Surface Wave Height at successive times (example 1) with numbers on each curve indicating the number of time steps Δt.

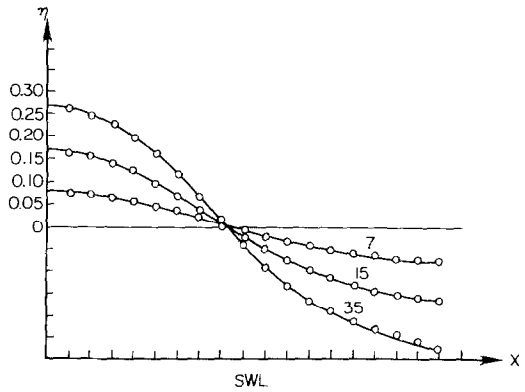

FIG. 3. Evolution of equilibrium free surface (SWL) with initial impulsive free surface velocity at successive times (examples 2) with numbers on each curve indicating the number of time steps Δt.

implicit system of equations (13). Loop (a) is embedded in Loop (b). The operational counts of Loop (a), ignoring $I/0$ and other data management time is $\sim M(5+\log M)$ per Fourier component, where M is the number of mesh spacings in the x-direction (and also ξ). Loop (a) requires Ka cycles to satisfy the relative convergence criterion. We have to repeat Loop (a) for each of the S Fourier components with $1 \leqslant s \leqslant S \leqslant M$, where the number S of the Fourier components used should not exceed the number of mesh spaces used in the computation. The operational counts of quasi-linear up-dating of ϕ and η from Eq. (13) is insignificant compared with $SM(5+\log M)$. Thus if Loop (b) requires Kb cycles for "convergence," the overall operational counts per time step is $\sim M(5+\log M)\cdot S\cdot Ka\cdot Kb$. Both Ka and Kb depend on the convergence criterion; and our choice of $<10^{-5}$ may be too restrictive in practical application. They also depend on the grid spacing ($\sim M^{-1}$) and the time step size satisfying the local CFL condition. As expected, our calculations suggest that reducing grid sizes promotes "convergence," i.e., smaller values of K, in particular Kb. There is clearly an optimum choice of M to minimize the overall computational effort for a given "accuracy" although we did not search for such an optimum. With $M \geqslant 20$, and $\Delta t = 0.02$ as is in the present examples, Ka is small ($1 \sim 2$) and $Kb \sim 5$ for example 1, and ~ 10 for example 2. The overall computational effort clearly varies appreciably for different types of initial-boundary data. For the types of initial-boundary data specified herein, the proposed discrete method of solution of such free surface wave problems is very efficient, not only in operational counts per iteration, but in overall computational effort to fulfill some preassigned criterion of iterative convergence.

DISCUSSION

We have developed an Eulerian method for the computational solution of the temporal evolution of potential free surface flows in two space dimensions. The

method is stable and efficient without any need of smoothing and filtering. The free surface condition is implemented in its full nonlinear form.

The method is equally applicable to more general situations, with arbitrary surface disturbances that are Fourier decomposable. The bottom surface need not be flat if the equilibrium depth of the fluid d is given as some known function of x. The solid walls may be replaced by imaginary surfaces with inflow or outflow velocity specified in terms of ϕ_n. The computational formulation remains the same except possibly for some changes in the algebraic details.

The extension of the present formulation from two to three space dimensions is straightforward. The wave equations (2) and (3) for the free surface will then be solved for $y=(x, z; t)$ in two space dimensions. So long as the Poisson problem (Eq. (13)) is posed properly for the solution of the velocity potential function $\phi(x, y, z; t)$, the computational solution for the free surface problem in 3D and 2D will remain essentially the same. The actual solution of such 3D problems will need, of course, significantly larger and faster computer, and adequate fast Poisson solver in 3 space to secure reasonable iterative convergence of the computation. We note that the straight forward extension of many 2D Fast Poisson Solver to 3D is prone to spurious oscillations (or even computational instability). If the present method is simply extended by using FFT in both x and z directions, we observed some mild spurious oscillations, removable by simple algebraic smoothing every 5 to 10 iterations. The computation is still quite efficient although the result is less certain vis a vis the 2D problems. Development of adequate fast Poisson solver in 3D can be quite profitable and may be necessary.

In the analysis of free surface problems of practical interest, the disturbed initial location $\eta_0(x)$ of the free surface is expected to be given but the associated free surface potential $\phi(x, y=\eta; t=0)=g(x)$ is unlikely known. Other data will have to be supplied instead. Whether such alternative specification of the initial-boundary data will render a well posed problem for solution as described in previous section. is uncertain.

When the initial speed of receding (or rising) of the free surface $\eta_t(x, t=0)$ is given along with $\eta_0(x)$, we may solve the Poisson Equation (8a) to obtain $\phi(x, y=\eta; t=0)$ in the following manner. Firstly, we transform the initial data η_t into some boundary data along $\xi=1$ (i.e., $y=\eta$) to complement the boundary conditions (8d) and (8e) specified on the remaining portion of $\partial\Omega$. With η_t and $\eta_x=\partial/\partial x\ \eta_0(x)$ given initially, Eq. (3) specifies a linear relation between ϕ_x and ϕ_y. When transformed into (x, ξ) coordinate it becomes a linear relation between the spatial derivatives ϕ_x along $\eta_0(x)$ and ϕ_ν along the inward normal ν of the free surface, i.e., $\phi_\nu=\alpha\phi_x+\beta$ on $\xi=1$. We can also solve iteratively Eq. (8a) as a succession of Neumann problems by first guessing and then iteratively correcting the ϕ_x from the previous iterants. The converged results gives $\phi(x, \xi, t=0)$ with ϕ_x at $\xi=1$ sufficiently close to that used in evaluating the ϕ_ν of the previous iterant. This converged $\phi(x, \xi; t=0)$ provides the needed initial data, $\phi(x, \xi=1; t=0)$ and the ϕ_x and ϕ_y on $\xi=1$ (or $\eta=\eta_0(x)$), for the determination of free surface wave development at later times as has been described previously.

When neither ϕ nor η_t on the free surface $y = \eta$ are available as the initial data, it is difficult to find suitable alternative on the free surface to provide boundary data along the entire closed boundary $\partial\Omega$, needed for the solution of $\phi(x, y; t)$ in Ω from $\nabla^2\phi = 0$. We are skeptical if some pressure data away from the free surface might be used instead. This is because the specification of pressure in addition to the local Neumann condition (8d) or (8e) creates a situation analogous to the Cauchy data which renders the Poisson problem ill-posed. Further investigation is needed to clarify such alternative formulation(s) to facilitate the solution of many free surface problems encountered in practice.

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